TOPOLOGICAL PROPERTIES OF PUNCTUAL HILBERT SCHEMES OF ALMOST-COMPLEX FOURFOLDS (I)

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Abstract. In this article, we study topological properties of Voisin's punctual Hilbert schemes of an almost-complex fourfold X. In this setting, we compute their Betti numbers and construct Nakajima operators. We also define tautological bundles associated with any complex bundle on X, which are shown to be canonical in K-theory.

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2. Introduction

Our aim in this paper is to extend some properties of the cohomology of punctual Hilbert schemes on smooth projective surfaces to the case of almost-complex compact manifolds of dimension four.

Let X be a smooth complex projective surface. For any integer $n \in \mathbb{N}^*$, the punctual Hilbert scheme $X^{[n]}$ is defined as the set of all 0-dimensional subschemes of X of length n. A theorem of Fogarty [Fo] states that $X^{[n]}$ is a smooth irreducible projective variety of complex dimension $2n$. The Hilbert-Chow map $HC : X^{[n]} \longrightarrow S^nX$ defined by $HC(\xi) = \sum_{x \in \mathrm{supp}(\xi)} \ell_x(\xi) x$ is a desingularization of the symmetric product S^nX. This implies that the varieties $X^{[n]}$ can be seen as smooth compactifications of the sets of distinct unordered n-tuples of points in X.

Voisin constructed in [Vo 1] punctual Hilbert schemes $X^{[n]}$ when X is only supposed to be a smooth almost-complex compact fourfold. This construction produces almost-complex Hilbert schemes $X^{[n]}$ which are differentiable manifolds of dimension $4n$ endowed with a stable almost-complex structure. Moreover there exists a continuous Hilbert-Chow map $HC : X^{[n]} \longrightarrow S^nX$ whose fibers are homeomorphic to the fibers of the Hilbert-Chow map in the integrable case.

Using ideas of Voisin concerning relative integrable structures, we generalize to the almost-complex setting some results already known in the integrable case. In this paper, we will mainly focus on the additive structure of the cohomology groups of $X^{[n]}$ with rational coefficients. We first expose Voisin's construction and we study the local topological structure of the Hilbert-Chow map. This allows us to compute the Betti Numbers of $X^{[n]}$, thus extending Göttsche's formula [Gö], [Gö-So] to the almost-complex case.

Theorem 1. *Let (X, J) be an almost-complex compact fourfold and, for any positive integer n, let $\big(b_i(X^{[n]})\big)_{i=0,\dots,4n}$ be the sequence of Betti numbers of the almost-complex Hilbert scheme $X^{[n]}$. Then the generating function for these Betti numbers is given by the formula*

$$\sum_{n=0}^{\infty}\sum_{i=0}^{4n} b_i\big(X^{[n]}\big)t^i q^n = \prod_{m=1}^{\infty}\frac{\big[(1+t^{2m-1}q^m)(1+t^{2m+1}q^m)\big]^{b_1(X)}}{(1-t^{2m-2}q^m)(1-t^{2m+2}q^m)(1-t^{2m}q^m)^{b_2(X)}}.$$

The proof of Theorem 1 relies on a topological version of the decomposition theorem of [De-Be-Be-Ga] for semi-small maps, which is due to Le Potier [LP].

The second part of the paper is devoted to the definition and the study of the Nakajima operators $q_i(\alpha)$ of an arbitrary almost-complex compact fourfold X. These operators are obtained as actions by correspondence of incidence varieties, constructed in the almost-complex setting. The incidence varieties are stratified topological spaces locally modelled on analytic spaces. We prove in this context the Nakajima commutation relations [Na]:

Theorem 2. *For any pair (i, j) of integers and any pair (α, β) of cohomology classes in $H^*(X, \mathbb{Q})$ we have*

$$\big[\mathfrak{q}_i(\alpha), \mathfrak{q}_j(\beta)\big] = i\,\delta_{i+j,0}\left(\int_X \alpha\beta\right)\mathrm{id}\,.$$

It follows from Theorems 1 and 2 that the Nakajima operators produce an irreducible representation of the Heisenberg super-algebra $\mathcal{H}(H^*(X,\mathbb{Q}))$ on $\bigoplus_{n\in\mathbb{N}} H^*(X^{[n]},\mathbb{Q})$.

In the last part, we explain how to construct tautological complex bundles $E^{[n]}$ on the almost-complex Hilbert schemes $X^{[n]}$ starting from a complex vector bundle E on X. To do so, we use variable holomorphic structures on E in the same spirit as the variable holomorphic structures on X used in Voisin's construction to define $X^{[n]}$.

If X is projective, Nakajima's theory as well as the tautological bundles are the fundamental tools to describe the ring structure of $H^*(X^{[n]},\mathbb{Q})$ (see [Le]). In a forthcoming paper, we will use the analogous almost-complex objects we have constructed here to compute the ring structure of $H^*(X^{[n]},\mathbb{Q})$ when X is a compact symplectic fourfold with vanishing first Betti number.

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3. The Hilbert scheme of an almost-complex compact fourfold

3.1. **Voisin's construction.** In this section, we recall Voisin's construction of the almost-complex Hilbert scheme and establish some complementary results. Let (X, J) be an almost-complex compact fourfold. The symmetric product S^nX will be endowed with the sheaf $\mathcal{C}^\infty_{S^nX}$ of $\mathcal{C}^\infty$ functions on X^n invariant under $\mathfrak{S}_n$. Let us introduce the incidence set

$$Z_n = \{(\underline{x}, p) \in S^nX \times X, \text{ such that } p \in \underline{x}\}. \tag{1}$$

Definition 3.1. For $\varepsilon > 0$, let $\mathcal{B}_\varepsilon$ be the set of pairs $(W, J^{\,\mathrm{rel}})$ such that

(i) W is a neighbourhood of Z_n in $S^nX \times X$,

(ii) $J^{\,\mathrm{rel}}$ is a relative integrable complex structure on the fibers of $\mathrm{pr}_1 : W \longrightarrow S^nX$ depending smoothly on the parameter $\underline{x}$ in S^nX,

(iii) if g is a fixed riemannian metric on X, $\sup_{\underline{x}\in S^nX,\, p\in W_{\underline{x}}} ||J^{\,\mathrm{rel}}_{\underline{x}}(p) - J_{\underline{x}}(p)||_{\,g} \leq \varepsilon$.

For ε small enough, $\mathcal{B}_\varepsilon$ is connected and weakly contractible (i.e. $\pi_i(\mathcal{B}_\varepsilon) = 0$ for $i \geq 1$). We choose such a ε and write $\mathcal{B}$ instead of $\mathcal{B}_\varepsilon$.

Let $\pi : (W^{[n]}_{\mathrm{rel}}, J^{\,\mathrm{rel}}) \longrightarrow S^nX$ be the relative Hilbert scheme of $(W, J^{\,\mathrm{rel}})$ over S^nX. The fibers of π are the smooth analytic sets $(W^{[n]}_{\underline{x}}, J^{\,\mathrm{rel}}_{\underline{x}})$, $\underline{x} \in S^nX$. Let $HC_{\mathrm{rel}} : W^{[n]}_{\mathrm{rel}} \longrightarrow S^n_{\mathrm{rel}}W$ be the relative Hilbert-Chow morphism over S^nX.

Definition 3.2. The *topological Hilbert scheme* $X^{[n]}_{J^{\,\mathrm{rel}}}$ is $(\pi, \mathrm{pr}_2 \circ HC_{\mathrm{rel}})^{-1}(\Delta_{S^nX})$, where Δ_{S^nX} is the diagonal of S^nX. More explicitely,

$$X^{[n]}_{J^{\,\mathrm{rel}}} = \{(\xi, \underline{x}) \text{ such that } \underline{x} \in S^nX,\ \xi \in (W^{[n]}_{\underline{x}}, J^{\,\mathrm{rel}}_{\underline{x}}) \text{ and } HC(\xi) = \underline{x}\}.$$

To put a differentiable structure on $X^{[n]}_{J^{\,\mathrm{rel}}}$, Voisin uses specific relative integrable structures which are invariant by a compatible system of retractions on the strata of S^nX. These relative structures are differentiable for a differentiable structure $\mathfrak{D}_J$ on S^nX which depends on J and is weaker than the quotient differentiable structure, i.e. $\mathfrak{D}_J \subseteq \mathcal{C}^\infty_{S^nX}$. The main result of Voisin is the following:

Theorem 3.3. [Vo 1], [Vo 2]

(i) $X^{[n]}$ *is a* $4n$*-dimensional differentiable manifold, well-defined modulo diffeomorphisms homotopic to identity.*

(ii) *The Hilbert-Chow map* $HC : X^{[n]} \longrightarrow (S^nX, \mathfrak{D}_J)$ *is differentiable and its fibers* $HC^{-1}(\underline{x})$ *are homeomorphic to the fibers of the usual Hilbert-Chow morphism for any integrable structure near* $\underline{x}$.

(iii) $X^{[n]}$ *can be endowed with a stable almost-complex structure, hence defines an almost-complex cobordism class. When* X *is symplectic,* $X^{[n]}_{J^{\,\mathrm{rel}}}$ *is symplectic.*

The first point is the analogue of Fogarty's result [Fo] in the differentiable case. In this article we will not use differentiable properties of $X^{[n]}$ but only topological ones, which allows us to

work with $X^{[n]}_{J^{\,\mathrm{rel}}}$ for any $J^{\,\mathrm{rel}}$ in $\mathcal{B}$. Without any assumption on $J^{\,\mathrm{rel}}$, the point (i) in Theorem 3.3 has the following topological version:

Proposition 3.4. *If $J^{\,\mathrm{rel}}\in \mathcal{B}$, $X^{[n]}_{J^{\,\mathrm{rel}}}$ is a $4n$-dimensional topological manifold.*

Proof. Let $\underline{x}_0 \in S^nX$. There exist holomorphic relative coordinates $(z_{\underline{x}}, w_{\underline{x}})$ for $J^{\,\mathrm{rel}}_{\underline{x}}$ in a neighbourhood of $\underline{x}_0$ which depend smoothly on $\underline{x}$. For every $\underline{x}$ near $\underline{x}_0$, the map $p \longrightarrow (z_{\underline{x}}(p), w_{\underline{x}}(p))$ is a biholomorphism between $(W_{\underline{x}}, J^{\,\mathrm{rel}}_{\underline{x}})$ and its image in $\mathbb{C}^2$ with the standard complex structure. Let us write $(z_{\underline{x}}(p), w_{\underline{x}}(p)) = \phi(x,p)$, where ϕ is a smooth function defined for x near a lift x_0 of $\underline{x}_0$, invariant under the action of the stabilizer of x_0 in $\mathfrak{S}_n$.

We write $x_0 = (y_1, \dots, y_1, \dots, y_k, \dots, y_k)$ where the points y_j are pairwise distinct and each y_j appears n_j times. We will identify small distinct neighbourhoods of y_j in X with distinct balls $B(y_j, \varepsilon)$ in $\mathbb{C}^2$. ϕ is defined on $B(y_1,\varepsilon)^{n_1} \times \cdots \times B(y_k, \varepsilon)^{n_k} \times \cup_{j=1}^k B(y_j, \varepsilon)$ and is $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_k}$ invariant. We can also suppose that $\phi(x_0, \,.\,) = \mathrm{id}$. We introduce new holomorphic coordinates by the formula $\widetilde{\phi}(x,p) = \phi(x,p) - D_1\, \phi(x_0, y_j)(x - x_0)$ if $p \in B(y_j, \varepsilon)$. Let

$$\Gamma\colon B(y_1,\varepsilon)^{n_1} \times \cdots \times B(y_k,\varepsilon)^{n_k} \longrightarrow \left(\mathbb{C}^2\right)^n$$

be defined by

$$\Gamma(x_1, \dots, x_n) = (\widetilde{\phi}(x_1, \dots, x_n, x_1), \dots, \widetilde{\phi}(x_1, \dots, x_n, x_n)).$$

The map Γ is $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_k}$ equivariant and has for differential at x_0 the identity map, so it induces a local homeomorphism γ of S^nX around $\underline{x}_0$. The image of the chart of $(X^{[n]}, J^{\,\mathrm{rel}})$ over a neighbourhood of $\underline{x}_0$ will be the classical Hilbert scheme $(\mathbb{C}^2)^{[n]}$ over a neighbourhood of $\underline{x}_0$. The chart and its inverse are given by the formulae $\varphi(\xi) = \phi(\underline{x}, \,.\,)_*$, where $HC(\xi) = \underline{x}$, and $\varphi^{-1}(\eta) = (\phi(\underline{y}, \,.\,)^{-1})_*\eta$, where $\underline{y} = \gamma^{-1}(HC(\eta))$. □

Remark 3.5. Let $J^{\,\mathrm{rel}}_0$ and $J^{\,\mathrm{rel}}_1$ be two relative integrable complex structures, and let ϕ_0, ϕ_1, γ_0 and γ_1 be defined as above. Then $X^{[n]}_{J_0^{\,\mathrm{rel}}}$ and $X^{[n]}_{J_1^{\,\mathrm{rel}}}$ are homeomorphic over a neighbourhood of $\underline{x}_0$. If $\phi(\underline{x}, p) = \phi_1^{-1}(\gamma_1^{-1}\gamma_0(\underline{x}), \phi_0(\underline{x}, p))$ and $\gamma(\underline{x}) = \gamma_1^{-1}\gamma_0(\underline{x})$, then there is a commutative diagram

$$\begin{tikzcd}
X^{[n]}_{J_1^{\,\mathrm{rel}}} \arrow[d, "HC"] & HC^{-1}(V_{\underline{x}_0}) \arrow[l, hook'] \arrow[r, "\phi_*", "\sim"'] \arrow[d] & HC^{-1}(\widetilde{V}_{\underline{x}_0}) \arrow[r, hook] \arrow[d] & X^{[n]}_{J_1^{\,\mathrm{rel}}} \arrow[d, "HC"] \\
S^nX \supseteq & V_{\underline{x}_0} \arrow[r, "\gamma", "\sim"'] & \widetilde{V}_{\underline{x}_0} & \subseteq S^nX
\end{tikzcd}$$

and γ is a stratified isomorphism.

3.2. **Göttsche's formula.** We will now turn our attention to the cohomology of $X^{[n]}_{J^{\,\mathrm{rel}}}$. The first step is the computation of the Betti numbers of $X^{[n]}$. We first recall the proof of Göttsche and Soergel ([Gö-So]) and then we show how to adapt it in the non-integrable case.

Let X and Y be irreducible algebraic complex varieties and $f : Y \longrightarrow X$ be a proper morphism. We assume that X is stratified in such a way that f is a topological fibration over each stratum X_ν. We denote by d_ν the real dimension of the largest irreducible component of the fiber. If $Y_\nu = f^{-1}(X_\nu)$, $\mathcal{L}_\nu = R^{d_\nu} f_* \mathbb{Q}_{Y_\nu}$ will be the associated monodromy local system on X_ν.

Definition 3.6. – The map f is a *semi-small morphism* if for all ν, $\operatorname{codim}_X X_\nu \geq d_\nu$.
– A stratum X_ν is *essential* if $\operatorname{codim}_X X_\nu = d_\nu$.

Theorem 3.7. [De-Be-Be-Ga] *If Y is rationally smooth and $f : Y \longrightarrow X$ is a proper semi-small morphism, there exists a canonical quasi-isomorphism $Rf_*\mathbb{Q}_Y \xrightarrow{\sim} \bigoplus_{\nu \text{ essential}} j_{\nu *} IC^\bullet_{\overline{X}_\nu}(\mathcal{L}_\nu)[-d_\nu]$ in the bounded derived category of $\mathbb{Q}$-constructible sheaves on X, where $IC^\bullet_{\overline{X}_\nu}(\mathcal{L}_\nu)$ is the intersection complex on $\overline{X}_\nu$ associated to the monodromy local system $\mathcal{L}_\nu$ and $j_\nu : \overline{X}_\nu \longrightarrow X$ is the inclusion. In particular, $H^k(Y, \mathbb{Q}) = \bigoplus_{\nu \text{ essential}} IH^{k-d_\nu}(\overline{X}_\nu, \mathcal{L}_\nu)$.*

Remark 3.8. A simple proof of Theorem 3.7 is done in [LP] and can be found in [Gri]. Furthermore, this proof shows that $Rf_*\mathbb{Q}_Y \simeq \bigoplus_{\nu \text{ essential}} j_{\nu *} IC^\bullet_{\overline{X}_\nu}(\mathcal{L}_\nu)[-d_\nu]$ under the following weaker topological hypotheses: Y is a rationally smooth topological space, X is a stratified topological space and $f : Y \longrightarrow X$ is a proper map which is locally equivalent over X to a semi-small map between complex analytic varieties.

If X is a quasi-projective surface, the Hilbert-Chow morphism is semi-small with irreducible fibers (see [Br]), so that the monodromy local systems are trivial, and $X^{[n]}$ is smooth. The decomposition theorem gives Göttsche's formula for $b_i(X^{[n]})$. We now show that Göttsche's formula holds more generally for almost-complex Hilbert schemes.

Theorem 3.9 (Göttsche's formula)**.** *If (X, J) is an almost-complex compact fourfold, then for any integrable complex structure J^{rel},*

$$\sum_{n=0}^{\infty} \sum_{i=0}^{4n} b_i\big(X^{[n]}_{J^{\text{rel}}}\big) t^i q^n = \prod_{m=1}^{\infty} \frac{\big[(1+t^{2m-1}q^m)(1+t^{2m+1}q^m)\big]^{b_1(X)}}{(1-t^{2m-2}q^m)(1-t^{2m+2}q^m)(1-t^{2m}q^m)^{b_2(X)}}.$$

Proof. By Remark 3.8, it suffices to check that $HC : X^{[n]}_{J^{\text{rel}}} \longrightarrow S^n X$ is locally equivalent to a semi-small morphism. The proof of Proposition 3.4 shows that $HC : X^{[n]}_{J^{\text{rel}}} \longrightarrow S^n X$ is locally equivalent to $HC : U^{[n]} \longrightarrow S^{[n]} U$ where U is an open set in $\mathbb{C}^2$. Thus the decomposition theorem applies and the computations are the same as in the integrable case. □

3.3. **Variation of the relative integrable structure.** Theorem 3.9 implies in particular that the Betti numbers of $X^{[n]}_{J^{\text{rel}}}$ are independent of J^{rel}. We now prove a stronger result, namely that the Hilbert schemes corresponding to different relative integrable complex structures are homeomorphic.

Proposition 3.10.

(i) *Let $\big(J^{\text{rel}}_t\big)_{t \in B(0,r) \subseteq \mathbb{R}^d}$ be a smooth path in $\mathcal{B}$. Then the associated relative Hilbert scheme $\big(X^{[n]}, \{J^{\text{rel}}_t\}_{t \in B(0,r)}\big)$ over $B(0,r)$ is a topological fibration.*

(ii) *If J^{rel}_0, $J^{\text{rel}}_1 \in \mathcal{B}$, then there exist canonical isomorphisms $H^*\big(X^{[n]}_{J^{\text{rel}}_0}, \mathbb{Q}\big) \simeq H^*\big(X^{[n]}_{J^{\text{rel}}_1}, \mathbb{Q}\big)$ and $K\big(X^{[n]}_{J^{\text{rel}}_0}\big) \simeq K\big(X^{[n]}_{J^{\text{rel}}_1}\big)$.*

In order to prove Proposition 3.10, we first establish the following result:

Proposition 3.11. *Let* $\big(J_t^{\text{rel}}\big)_{t\in B(0,r)\subseteq\mathbb{R}^d}$ *be a family of smooth relative complex structures in a neighbourhood of* Z_n *varying smoothly with* t*. Then there exist* $\varepsilon>0$*, a neighbourhood* W *of* Z_n *in* $S^nX\times X$ *and a continuous map* $\psi:(t,\underline{x},p)\longmapsto\psi_{t,\underline{x}}(p)$ *from* $B(0,\varepsilon)\times W$ *to* X *such that:*

(i) $\psi_{0,\underline{x}}(p)=p$,

(ii) *For fixed* $(t,\underline{x})$, $\psi_{t,\underline{x}}$ *is a biholomorphism between a neighbourhood of* $\underline{x}$ *and a neighbourhood of* $S^n\psi_{t,\underline{x}}(\underline{x})$*, endowed with the structures* $J_{0,\underline{x}}^{\text{rel}}$ *and* $J_{t,\psi_{t,\underline{x}}(\underline{x})}^{\text{rel}}$,

(iii) *For all* t *in* $B(0,\varepsilon)$*, the map* $\underline{x}\longrightarrow S^n\psi_{t,\underline{x}}(\underline{x})$ *is a homeomorphism of* S^nX.

Proof. We can choose a family of maps θ_t varying smoothly with t such that for all $\underline{x}$ in S^nX and t in $B(0,r)$, $\theta_{t,\underline{x}}$ is a biholomorphism between two neighbourhoods of $\underline{x}$ endowed with the structures $J_{t,\underline{x}}^{\text{rel}}$ and $J_{0,\underline{x}}^{\text{rel}}$, and such that for all $\underline{x}$ in S^nX, $\theta_{0,\underline{x}}=\text{id}$. We take, as in the proof of Proposition 3.4, a system $\big(\phi_{\underline{x}}^i\big)_{1\le i\le N}$ of holomorphic relative coordinates for J_0^{rel} with respect to a covering $\big(\widetilde{U}_i\big)_{1\le i\le N}$ of S^nX such that $\underline{x}\longrightarrow S^n\phi_{\underline{x}}^i(\underline{x})$ is a homeomorphism between $\widetilde{U}_i$ and its image $\widetilde{V}_i$ in $S^n\mathbb{C}^2$. We define holomorphic relative coordinates $\big(\phi_{t,\underline{x}}^i\big)_{1\le i\le N}$ for J_t^{rel} by the formula $\phi_{t,\underline{x}}^i(p)=\phi_{\underline{x}}^i\big(\theta_{t,\underline{x}}(p)\big)$. For small t, after shrinking $\widetilde{U}_i$ if necessary, $\underline{x}\longrightarrow S^n\phi_{t,\underline{x}}^i(\underline{x})$ is still a homeomorphism: indeed the map $\underline{x}\longrightarrow S^n\phi_{t,\underline{x}}^i(\underline{x})$ is obtained from the $\mathfrak{S}_{n_1}\times\cdots\times\mathfrak{S}_{n_k}$ equivariant smooth map $(x_1,\dots,x_n)\longrightarrow\big(\phi_t(x_1,\dots,x_n,x_1),\dots,\phi_t(x_1,\dots,x_n,x_n)\big)$. Then we use the fact that a sufficiently small smooth perturbation of a smooth diffeomorphism remains a smooth diffeomorphism.

Let $\widetilde{Z}_n\subseteq S^n\mathbb{C}^2\times\mathbb{C}^2$ be the incidence variety of $\mathbb{C}^2$. The map $\check{\phi}_t^i:(\underline{x},p)\longrightarrow\big(S^n\phi_{t,\underline{x}}^i(\underline{x}),\phi_{t,\underline{x}}^i(p)\big)$ is a homeomorphism between two neighbourhoods of Z_n and $\widetilde{Z}_n$ over $\widetilde{U}_i$ and $\widetilde{V}_i$. If we define $\check{\psi}_t:(\underline{x},p)\longrightarrow\big(S^n\psi_{t,\underline{x}}(\underline{x}),\psi_{t,\underline{x}}(p)\big)$, the condition (ii) of the proposition means that $\check{\phi}_t^i\circ\check{\psi}_t\circ\big(\check{\phi}_0^i\big)^{-1}$ is of the form $(\underline{y},p)\longrightarrow\big(S^nu_{t,\underline{y}}(\underline{y}),u_{t,\underline{y}}(p)\big)$ where $\underline{y}\in\widetilde{V}_i$ and $u_{t,\underline{y}}$ is a biholomorphism between a neighbourhood of $\underline{y}$ and its image (both endowed with the standard complex structure of $\mathbb{C}^2$), varying smoothly with t and $\underline{y}$. The condition (i) means that $u_{0,\underline{y}}=\text{id}$. Thus $\big(\psi_t\big)_{||t||\le\varepsilon}$ can be constructed on small open sets of S^nX. Since biholomorphisms close to identity form a contractible set, we can, using cut-off functions, glue together the local solutions on S^nX to obtain a global one. The map $\underline{x}\longrightarrow S^n\psi_{t,\underline{x}}(\underline{x})$ is induced by a smooth $\mathfrak{S}_n$-equivariant map of X^n into X^n (and is a small perturbation of the identity map if $||t||$ is small enough), thus a $\mathfrak{S}_n$-equivariant diffeomorphism of X^n. We have therefore defined a family of maps $\big(\psi_t\big)_{||t||\le\varepsilon}$ satisfying the conditions (i), (ii) and (iii). $\square$

We can now prove Proposition 3.10.

Proof of Proposition 3.10. (i) We have
$\big(X^{[n]},\{J_t^{\mathrm{rel}}\}_{t\in B(0,r)}\big)=\Big\{(\xi,\underline{x},t)$ such that $\underline{x}\in S^nX,\ t\in B(0,r),\ \xi\in\big(W_{\underline{x}}^{[n]},J_{t,\underline{x}}^{\mathrm{rel}}\big),\ HC(\xi)=\underline{x}\Big\}\cdot$
A topological trivialization of this family over $B(0,r)$ near zero is given by the map

$$\Gamma\colon X^{[n]}_{J_0^{\mathrm{rel}}}\times B(0,\varepsilon)\longrightarrow\big(X^{[n]},\{J_t^{\mathrm{rel}}\}_{t\in B(0,\varepsilon)}\big)$$

defined by $\Gamma(\xi,\underline{x},t)=\big(\psi_{t,\underline{x}*}\xi,\psi_{t,\underline{x}}(\underline{x}),t\big)$, where ψ is given by Proposition 3.11. This proves that the relative Hilbert scheme is locally topologically trivial over $B(0,r)$.

(ii) The set $\mathcal{B}$ being connected, point (i) shows that $X^{[n]}_{J_0^{\mathrm{rel}}}$ and $X^{[n]}_{J_1^{\mathrm{rel}}}$ are homeomorphic. Since $\pi_1(\mathcal{B})=0$, if we consider two paths $\big(J_{0,t}^{\mathrm{rel}}\big)_{0\leq t\leq 1}$ and $\big(J_{1,t}^{\mathrm{rel}}\big)_{0\leq t\leq 1}$ between J_0^{rel} and J_1^{rel}, we can find a smooth family $\big(J_{s,t}^{\mathrm{rel}}\big)_{\substack{0\leq s\leq 1\\ 0\leq t\leq 1}}$ which is an homotopy between the two paths. The relative associated Hilbert scheme over $[0,1]\times[0,1]$ is locally topologically trivial, hence globally trivial since $[0,1]\times[0,1]$ is contractible. This shows that the homeomorphisms between $X^{[n]}_{J_0^{\mathrm{rel}}}$ and $X^{[n]}_{J_1^{\mathrm{rel}}}$ constructed by the procedure above belong to a canonical homotopy class. $\square$

As a consequence of this proposition, there exists a ring $H^*(X^{[n]},\mathbb{Q})$ (resp. $K(X^{[n]})$) such that for any J^{rel} close to J, $H^*(X^{[n]},\mathbb{Q})$ (resp. $K(X^{[n]})$) and $H^*\big(X^{[n]}_{J^{\mathrm{rel}}},\mathbb{Q}\big)$ (resp. $K\big(X^{[n]}_{J^{\mathrm{rel}}}\big)$) are canonically isomorphic.

In the sequel, we will deal with products of Hilbert schemes. We can of course consider products of the type $\big(X^{[n]}_{J_n^{\mathrm{rel}}}\big)\times\big(X^{[m]}_{J_m^{\mathrm{rel}}}\big)$, but in practice it is necessary to work with pairs of relative integrable complex structures parametrized by $(\underline{x},\underline{y})$ in $S^nX\times S^mX$. Let us introduce the incidence set

$$Z_{n\times m}=\big\{(\underline{x},\underline{y},p)\text{ in }\big(S^nX\times S^mX\big)\times X\text{ such that }p\in\underline{x}\cup\underline{y}\big\}\cdot\tag{2}$$

Let W be a small neighbourhood of $Z_{n\times m}$ and let $J^{1,\mathrm{rel}}$ and $J^{2,\mathrm{rel}}$ be two relative integrable complex structures on the fibers of $\mathrm{pr}_1: W\longrightarrow S^nX\times S^mX$.

Definition 3.12. The product Hilbert scheme $\big(X^{[n]\times[m]},J^{1,\mathrm{rel}},J^{2,\mathrm{rel}}\big)$ is defined by

$$\big(X^{[n]\times[m]},J^{1,\mathrm{rel}},J^{2,\mathrm{rel}}\big)=\Big\{(\xi,\eta,\underline{x},\underline{y})\text{ such that }\underline{x}\in S^nX,\ \underline{y}\in S^mX,\ \xi\in\big(W^{[n]}_{\underline{x},\underline{y}},J^{1,\mathrm{rel}}_{\underline{x},\underline{y}}\big),\\ \eta\in\big(W^{[m]}_{\underline{x},\underline{y}},J^{2,\mathrm{rel}}_{\underline{x},\underline{y}}\big),\ HC(\xi)=\underline{x},\ HC(\eta)=\underline{y}\Big\}\cdot$$

The same definition holds for products of the type $\Big(X^{[n_1]\times\cdots\times[n_k]},J^{1,\mathrm{rel}},\dots,J^{k,\mathrm{rel}}\Big)$.

If there exist two relative integrable complex structures J_n^{rel} and J_m^{rel} in neighbourhoods of Z_n and Z_m such that $J^{1,\mathrm{rel}}_{\underline{x},\underline{y}}=J^{\mathrm{rel}}_{n,\underline{x}}$ and $J^{2,\mathrm{rel}}_{\underline{x},\underline{y}}=J^{\mathrm{rel}}_{m,\underline{x}}$ in small neighbourhoods of $\underline{x}$ and $\underline{y}$, we have

$$\big(X^{[n]\times[m]},J^{1,\mathrm{rel}},J^{2,\mathrm{rel}}\big)=X^{[n]}_{J_n^{\mathrm{rel}}}\times X^{[m]}_{J_m^{\mathrm{rel}}}.$$

If $\big(J_t^{1,\mathrm{rel}},J_t^{2,\mathrm{rel}}\big)_{t\in B(0,r)}$ is a smooth family of relative integrable complex structures, it can be shown as in Propositions 3.11 and 3.10 that the family $\Big(X^{[n]\times[m]},\{J_t^{1,\mathrm{rel}}\}_{t\in B(0,r)},\{J_t^{2,\mathrm{rel}}\}_{t\in B(0,r)}\Big)$ is topologically trivial over $B(0,r)$. Thus the product Hilbert schemes $\big(X^{[n]\times[m]},J^{1,\mathrm{rel}},J^{2,\mathrm{rel}}\big)$ is

isomorphic to products $X^{[n]}_{J^{\mathrm{rel}}_n}\times X^{[m]}_{J^{\mathrm{rel}}_m}$ of usual Hilbert schemes. If the structures $J^{1,\mathrm{rel}}$ and $J^{2,\mathrm{rel}}$ are equal, $\big(X^{[n]\times[m]},J^{1,\mathrm{rel}},J^{2,\mathrm{rel}}\big)$ consists of pairs of schemes of given support (parametrized by $S^nX\times S^mX$) for the *same* integrable structure. These product Hilbert schemes are therefore well adapted for the study of incidence relations.

4. Incidence varieties and Nakajima operators

4.1. **Construction of incidence varieties.** If J is an integrable complex structure on X, the *incidence variety* $X^{[n',n]}$ is classically defined by

$$X^{[n',n]}=\big\{(\xi,\xi')\text{ such that }\xi\in X^{[n]},\ \xi'\in X^{[n']}\text{ and }\xi\subseteq\xi'\big\}\cdot$$

The incidence variety $X^{[n',n]}$ is never smooth unless $n'=n+1$ (see [Ti]). We have three maps $\lambda:\,X^{[n',n]}\longrightarrow X^{[n]}$, $\nu:\,X^{[n',n]}\longrightarrow X^{[n']}$ and $\rho:\,X^{[n',n]}\longrightarrow S^{n'-n}X$ given by $\lambda(\xi,\xi')=\xi$, $\nu(\xi,\xi')=\xi'$ and $\rho(\xi,\xi')=\mathrm{supp}\big(\mathcal{I}_{\xi}/_{\mathcal{I}_{\xi'}}\big)$. Note that, by definition, (λ,ν) is injective.

If J is not integrable, we can define $X^{[n',n]}$ using the relative construction. Let $J^{\mathrm{rel}}_{n\times(n'-n)}$ be a relative integrable complex structure in a neighbourhood of $Z_{n\times(n'-n)}$ in $S^nX\times S^{n'-n}X\times X$ (see (2)).

Definition 4.1. The incidence variety $\big(X^{[n',n]},J^{\mathrm{rel}}_{n\times(n'-n)}\big)$ is defined by

$$\big(X^{[n',n]},J^{\mathrm{rel}}_{n\times(n'-n)}\big)=\Big\{(\xi,\xi',\underline{x},\underline{y})\text{ such that }\underline{x}\in S^nX,\ \underline{y}\in S^{n'-n}X,\ \xi\in\big(W^{[n]}_{\underline{x}},J^{\mathrm{rel}}_{n\times(n'-n),\,\underline{x},\,\underline{y}}\big),\\ \xi'\in\big(W^{[n']}_{\underline{x}\cup\underline{y}},J^{\mathrm{rel}}_{n\times(n'-n),\,\underline{x},\,\underline{y}}\big),\ \xi\subseteq\xi',\ HC(\xi)=\underline{x},\ \rho(\xi,\xi')=\underline{y}\Big\}\cdot$$

Let $J^{\mathrm{rel}}_{n\times n'}$ be a relative integrable complex structure in a neighbourhood of $Z_{n\times n'}$ such that for every $\underline{u}\in S^nX$ and $\underline{v}\in S^{n'-n}X$, $J^{\mathrm{rel}}_{n\times n',\underline{u},\underline{u}\cup\underline{v}}=J^{\mathrm{rel}}_{n\times(n'-n),\underline{u},\underline{v}}$. Then

$$\big(X^{[n',n]},J^{\mathrm{rel}}_{n\times(n'-n)}\big)\subseteq\big(X^{[n]\times[n']},J^{\mathrm{rel}}_{n\times n'},J^{\mathrm{rel}}_{n\times n'}\big).$$

If $\big\{J^{\mathrm{rel}}_{t,n\times n'}\big\}_{t\in B(0,r)}$ is a smooth family of relative complex structures, we can take, as in Proposition 3.10, a topological trivialization of $\big(X^{[n]\times[n']},\big\{J^{\mathrm{rel}}_{t,n\times n'}\big\}_{t\in B(0,r)},\big\{J^{\mathrm{rel}}_{t,n\times n'}\big\}_{t\in B(0,r)}\big)$. If we define $J^{\mathrm{rel}}_{t,n\times(n'-n)}$ in a neighbourhood of $Z_{n\times(n'-n)}$ by the formula $J^{\mathrm{rel}}_{t,n\times(n'-n),\underline{u},\underline{v}}=J^{\mathrm{rel}}_{t,\,n\times n',\underline{u},\,\underline{u}\cup\underline{v}}$, then we can choose the trivialization so that the subfamily $\big(X^{[n',n]},\big\{J^{\mathrm{rel}}_{t,n\times(n'-n)}\big\}_{t\in B(0,r)}\big)$ is sent to the product $U^{[n',n]}\times B(0,\varepsilon)$, where U is an open set of $\mathbb{C}^2$. This means that the family

$$\Big\{\Big(X^{[n',n]},\big\{J^{\mathrm{rel}}_{n\times(n'-n)}\big\}_{t\in B(0,r)}\Big),\Big(X^{[n]\times[n']},\big\{J^{\mathrm{rel}}_{n\times n'}\big\}_{t\in B(0,r)},\big\{J^{\mathrm{rel}}_{n\times n'}\big\}_{t\in B(0,r)}\Big)\Big\}$$

is locally, hence globally topologically trivial over $B(0,r)$.

The natural morphism from $\big(X^{[n',n]},J_{n\times(n'-n)}\big)$ to $S^nX\times S^{n'-n}X$ is locally equivalent over $S^nX\times S^{n'-n}X$ to the natural morphism $U^{[n',n]}\longrightarrow S^nU\times S^{n'-n}U$. This enables us to define a stratification on $X^{[n',n]}$ by patching together the analytic stratifications of a collection of $U_i^{[n',n]}$.

In this way, $\big(X^{[n',n]}, J^{\mathrm{rel}}_{n\times(n'-n)}\big)$ becomes a stratified CW-complex such that for each stratum S, $\dim(\overline{S}\backslash S) \leq \dim S - 2$. In particular, each stratum has a homology class.

Let us introduce the following notations:

(i) The inverse image of the small diagonal of S^nX by $HC\colon X^{[n]}_{J^{\mathrm{rel}}_n} \longrightarrow S^nX$ will be denoted by $\big(X^{[n]}_0, J^{\mathrm{rel}}_n\big)$.

(ii) The inverse image of the small diagonal of $S^{n'-n}X$ by $\rho\colon \big(X^{[n',n]}, J^{\mathrm{rel}}_{n\times(n'-n)}\big) \longrightarrow S^{n'-n}X$ will be denoted by $\big(X^{[n',n]}_0, J^{\mathrm{rel}}_{n\times(n'-n)}\big)$.

In the integrable case, $X^{[n',n]}_0$ is stratified by analytic sets $\big(Z_l\big)_{l\geq 0}$ defined by

$$Z_l = \big\{(\xi,\xi') \in X^{[n',n]}_0 \text{ such that if } x=\rho(\xi,\xi'),\ \ell_x(\xi)=l\big\}; \tag{3}$$

Z_0 is irreducible of complex dimension $n'+n+1$, and all the other Z_l have smaller dimensions (see [Le]). By the same argument as above, this stratification also exists in the almost complex case. We prove the topological irreducibility of Z_0 in the following lemma:

Lemma 4.2. *Let $\big[\,\overline{Z}_0\big]$ be the fundamental homology class of $\overline{Z}_0$. Then*

$$H_{2(n'+n+1)}\big(X^{[n',n]}_0, J^{\mathrm{rel}}_{n\times(n'-n)}, \mathbb{Z}\big) = \mathbb{Z}\,.\big[\,\overline{Z}_0\big].$$

Proof. It is enough to prove that the Borel-Moore homology group $H^{\mathrm{lf}}_{2(n'+n+1)}(Z_0,\mathbb{Z})$ is $\mathbb{Z}$, since all the remaining strata $\big(Z_l\big)_{l\geq 1}$ have dimensions smaller than $2(n'+n+1)-2$. Let

$$\widetilde{Z}_0 = \Big\{(\xi,\eta,\underline{x},p) \text{ such that } \underline{x}\in S^nX,\ p\in X,\ \xi\in\big(W^{[n]}_{\underline{x},\,(n'-n)p}, J^{\mathrm{rel}}_{n\times(n'-n),\,\underline{x},\,(n'-n)p}\big),\ HC(\xi)=\underline{x},$$
$$\eta\in\big(W^{[n'-n]}_{\underline{x},\,(n'-n)p}, J^{\mathrm{rel}}_{n\times(n'-n),\,\underline{x},\,(n'-n)p}\big),\ HC(\eta)=(n'-n)p\Big\}.$$

There is a natural inclusion $Z_0 \hookrightarrow \widetilde{Z}_0$ given by $\big(\xi,\xi',\underline{x},(n'-n)p\big) \longrightarrow (\xi,\xi'_{|p},\underline{x},p)$. Remark that $\widetilde{Z}_0$ is compact. Since $\dim(\widetilde{Z}_0\backslash Z_0) \leq 4n+2(n'-n-1) = 2(n'+n-1)$, it suffices to show that $H_{2(n'+n+1)}(\widetilde{Z}_0,\mathbb{Z}) = \mathbb{Z}$. $\widetilde{Z}_0$ is a product-type Hilbert scheme homeomorphic to $X^{[n]}_{J^{\mathrm{rel}}_n} \times \big(X^{[n'-n]}_0, J^{\mathrm{rel}}_{n'-n}\big)$. Since $\big(X^{[n'-n]}_0, J^{\mathrm{rel}}_{n'-n}\big)$ is by Briançon's theorem [Br] a topological fibration on X whose fiber is homeomorphic to an irreducible algebraic variety of complex dimension $n'-n-1$, we obtain the result. □

4.2. **Nakajima operators.** We are now going to construct Nakajima operators $q_n(\alpha)$ in the almost-complex context. If $n'>n$, let us define

$$Q^{[n',n]} = \overline{Z}_0 \subseteq \big(X^{[n]\times[n']}\times X, J^{\mathrm{rel}}_{n\times n'}, J^{\mathrm{rel}}_{n\times n'}\big)\times X, \tag{4}$$

where the map on the last coordinate is given by the relative residual morphism and Z_0 is defined by (3). Since the pair $\big(Q^{[n',n]}, X^{[n]\times[n']}\times X\big)$ is topologically trivial when $J^{\mathrm{rel}}_{n\times n'}$ varies, the cycle class $\big[Q^{[n',n]}\big] \in H_{2(n'+n+1)}\big(X^{[n]}\times X^{[n']}\times X,\mathbb{Z}\big)$ is independent of $J^{\mathrm{rel}}_{n\times n'}$.

Definition 4.3. Let $\alpha \in H^*(X,\mathbb{Q})$ and $j \in \mathbb{N}^*$. We define the Nakajima operators $\mathfrak{q}_j(\alpha)$ and $\mathfrak{q}_{-j}(\alpha)$ as follows:

$$\begin{array}{rccc} \mathfrak{q}_j(\alpha): & \bigoplus_{n\in\mathbb{N}} H^*(X^{[n]},\mathbb{Q}) & \longrightarrow & \bigoplus_{n\in\mathbb{N}} H^*(X^{[n+j]},\mathbb{Q}) \\ & \tau & \longmapsto & PD^{-1}\Big[\mathrm{pr}_{2*}\big(\big[Q^{[n+j,n]}\big]\cap(\mathrm{pr}_3^*\,\alpha\cup\mathrm{pr}_1^*\,\tau)\big)\Big] \end{array}$$

$$\begin{array}{rccc} \mathfrak{q}_{-j}(\alpha): & \bigoplus_{n\in\mathbb{N}} H^*(X^{[n+j]},\mathbb{Q}) & \longrightarrow & \bigoplus_{n\in\mathbb{N}} H^*(X^{[n]},\mathbb{Q}) \\ & \tau & \longmapsto & PD^{-1}\Big[\mathrm{pr}_{1*}\big(\big[Q^{[n+j,n]}\big]\cap(\mathrm{pr}_3^*\,\alpha\cup\mathrm{pr}_2^*\,\tau)\big)\Big] \end{array}$$

where pr_1, pr_2 and pr_3 are the projections from $X^{[n]}\times X^{[n+j]}\times X$ to each factor and PD is the Poincaré duality isomorphism between cohomology and homology. We also set $\mathfrak{q}_0(\alpha)=0$

Remark 4.4. Let $|\alpha|$ be the degree of α, then $q_j(\alpha)$ maps $H^i(X^{[n]},\mathbb{Q})$ to $H^{i+|\alpha|+2j-2}(X^{[n+j]},\mathbb{Q})$.

We now prove the following extension to the almost-complex case of Nakajima's theorem [Na]:

Theorem 4.5. *For i, $j\in\mathbb{Z}$ and α, $\beta\in H^*(X,\mathbb{Q})$ we have*

$$\mathfrak{q}_i(\alpha)\mathfrak{q}_j(\beta)-(-1)^{|\alpha|\,|\beta|}\mathfrak{q}_j(\beta)\mathfrak{q}_i(\alpha)=i\,\delta_{i+j,0}\left(\int_X \alpha\beta\right)\mathrm{id}$$

.

Proof. We adapt Nakajima's proof to our situation. The most interesting case is the computation of $[\mathfrak{q}_{-i}(\alpha),\mathfrak{q}_j(\beta)]$ when i and j are positive. We introduce the classes $\big[P_{ij}\big]$, resp. $\big[Q_{ij}\big]$ in

$$H_*\big(X^{[n]},\mathbb{Q}\big)\otimes H_*\big(X^{[n-i]},\mathbb{Q}\big)\otimes H_*\big(X^{[n-i+j]},\mathbb{Q}\big)\otimes H_*\big(X,\mathbb{Q}\big)\otimes H_*\big(X,\mathbb{Q}\big),\qquad\text{resp.}$$
$$H_*\big(X^{[n]},\mathbb{Q}\big)\otimes H_*\big(X^{[n+j]},\mathbb{Q}\big)\otimes H_*\big(X^{[n-i+j]},\mathbb{Q}\big)\otimes H_*\big(X,\mathbb{Q}\big)\otimes H_*\big(X,\mathbb{Q}\big),$$

as follows:

$$\big[P_{ij}\big]:=p_{13*}\Big[p_{124}^*\big[Q^{[n,n-i]}\big]\cdot p_{235}^*\big[Q^{[n-i+j,n-i]}\big]\Big],\quad\text{resp.}$$
$$\big[Q_{ij}\big]:=p_{13*}\Big[p_{124}^*\big[Q^{[n+j,n]}\big]\cdot p_{235}^*\big[Q^{[n+j,n-i+j]}\big]\Big],$$

where $Q^{[r,s]}$ is defined in (4). Then $\mathfrak{q}_j(\beta)\mathfrak{q}_{-i}(\alpha)$, resp. $\mathfrak{q}_{-i}(\alpha)\mathfrak{q}_j(\beta)$, is given by

$$\tau\longmapsto PD^{-1}\Big[\mathrm{pr}_{3*}\big(\big[P_{ij}\big]\cap\big(\mathrm{pr}_5^*\,\beta\cup\mathrm{pr}_4^*\,\alpha\cup\mathrm{pr}_1^*\,\tau\big)\big)\Big],\quad\text{resp.}$$

$$\tau\longmapsto PD^{-1}\Big[\mathrm{pr}_{3*}\big(\big[Q_{ij}\big]\cap\big(\mathrm{pr}_5^*\,\alpha\cup\mathrm{pr}_4^*\,\beta\cup\mathrm{pr}_1^*\,\tau\big)\big)\Big].$$

First we deform all the relative integrable complex structures into a single one parametrized by $S^nX\times S^{n-i}X\times S^{n-i+j}X\times S^2X$. Let $J^{\mathrm{rel}}_{n\times(n-i)\times(n-i+j)\times 2}$ be a relative integrable structure in a neighbourhood of $Z_{n\times(n-i)\times(n-i+j)\times 2}$, and let

$$Y=\Big(X^{[n]\times[n-i]\times[n-i+j]\times[1]\times[1]},J^{\mathrm{rel}}_{n\times(n-i)\times(n-i+j)\times 2}\leftarrow 5\text{ times}\Big)$$

be the product Hilbert scheme obtained by taking the same structure $J^{\mathrm{rel}}_{n\times(n-i)\times(n-i+j)\times 2}$ five times (see Definition 3.12), where $J^{\mathrm{rel}}_{n\times(n-i)\times(n-i+j)\times 2}$ is identified with its pullback by

$$\mu\colon S^nX\times S^{n-i}X\times S^{n-i+j}X\times X\times X\longrightarrow S^nX\times S^{n-i}X\times S^{n-i+j}X\times S^2X.$$

Then, via the canonical isomorphism

$$H_*\big(Y,\mathbb{Q}\big)\simeq H_*\big(X^{[n]},\mathbb{Q}\big)\otimes H_*\big(X^{[n-i]},\mathbb{Q}\big)\otimes H_*\big(X^{[n-i+j]},\mathbb{Q}\big)\otimes H_*\big(X,\mathbb{Q}\big)\otimes H_*\big(X,\mathbb{Q}\big),$$

since incidence varieties vary trivially in families, the class $p_{124}^*\big[Q^{[n,n-i]}\big]$ is the homology class of the cycle

$$A=\Big\{(\xi,\xi',\xi'',s,t)\text{ such that }\xi'\subseteq\xi\text{ and }\rho(\xi',\xi)=s\Big\}.$$

In the same way, $p_{235}^*\big[Q^{[n-i+j,n-i]}\big]$ is the homology class of the cycle

$$B=\Big\{(\xi,\xi',\xi'',s,t)\text{ such that }\xi'\subseteq\xi''\text{ and }\rho(\xi',\xi'')=t\Big\}.$$

We now study the intersection of the cycles A and B. Let $p\in A\cap B$. We choose relative holomorphic coordinates $\phi_{\underline{x},\underline{y},\underline{z},s,t}$ for $J^{\mathrm{rel}}_{n\times(n-i)\times(n-i+j)\times 2}$ such that

$$(\underline{x},\underline{y},\underline{z},s,t)\longmapsto\big(S^n\phi_{\underline{x},\underline{y},\underline{z},s,t}(\underline{x}),S^{n-i}\phi_{\underline{x},\underline{y},\underline{z},s,t}(\underline{y}),S^{n-i+j}\phi_{\underline{x},\underline{y},\underline{z},s,t}(\underline{z}),\phi_{\underline{x},\underline{y},\underline{z},s,t}(s),\phi_{\underline{x},\underline{y},\underline{z},s,t}(t)\big)$$

is a local homeomorphism. The associated map given by

$$(\xi,\underline{x},\xi',\underline{y},\xi'',\underline{z},s,t)\longmapsto\big(\phi_{\underline{x},\underline{y},\underline{z},s,t*}\xi,\phi_{\underline{x},\underline{y},\underline{z},s,t*}\xi',S^{n-i+j}\phi_{\underline{x},\underline{y},\underline{z},s,t*}\xi'',\phi_{\underline{x},\underline{y},\underline{z},s,t}(s),\phi_{\underline{x},\underline{y},\underline{z},s,t}(t)\big)$$

is a homeomorphism from a neighbourhood of p to its image in $(\mathbb{C}^2)^{[n]}\times(\mathbb{C}^2)^{[n-i]}\times(\mathbb{C}^2)^{[n-i+j]}\times\mathbb{C}^2\times\mathbb{C}^2$ which maps A and B to the classical cycles $p_{124}^{-1}Q^{[n,n-i]}$ and $p_{235}^{-1}Q^{[n-i+j,n-i]}$. In the integrable case, we know that in the open set $\{s\neq t\}$, $p_{124}^{-1}Q^{[n,n-i]}$ and $p_{235}^{-1}Q^{[n-i+j,n-i]}$ intersect generically transversally. Using relative holomorphic coordinates as above, this property still holds in our context. If $\big(A\cap B\big)_{s\neq t}=C_{ij}$, we can write $[A]\cdot[B]=\big[\,\overline{C_{ij}}\,\big]+\iota_*R$ where $\iota:Y_{\{s=t\}}\hookrightarrow Y$ is the natural injection and $R\in H_{2(2n-i+j+2)}\big(Y_{\{s=t\}},\mathbb{Q}\big)$. We can do the same in

$$Y'=\Big(X^{[n]\times[n+j]\times[n-i+j]\times[1]\times[1]},J^{\mathrm{rel}}_{n\times(n+j)\times(n-i+j)\times 2}\leftarrow 5\text{ times}\Big)$$

with the cycles A' and B' defined by

$$A'=\Big\{(\xi,\xi',\xi'',s,t)\text{ such that }\xi\subseteq\xi',\ \rho(\xi,\xi')=s\Big\}$$
$$B'=\Big\{(\xi,\xi',\xi'',s,t)\text{ such that }\xi''\subseteq\xi',\ \rho(\xi'',\xi')=t\Big\}.$$

We put $D_{ij}=\big(A'\cap B'\big)_{s\neq t}$. Then $[A']\cdot[B']=\big[\,\overline{D_{ij}}\,\big]+\iota'_*R'$, where $\iota':Y'_{\{s=t\}}\hookrightarrow Y'$ is the injection and $R'\in H_{2(2n-i+j+2)}\big(Y'_{\{s=t\}},\mathbb{Q}\big)$. The class R (resp. R') can be chosen supported in $A\cap B\cap Y_{\{s=t\}}$ (resp. in $A'\cap B'\cap Y'_{\{s=t\}}$).

The following lemma describes the situation outside the diagonal $\{s=t\}$.

Lemma 4.6. $p_{1345*}\Big(\big[\,\overline{C_{ij}}\,\big]\cap\big(\mathrm{pr}_5^*\,\beta\cup\mathrm{pr}_4^*\,\alpha\big)\Big)=(-1)^{|\alpha|\,|\beta|}p_{1345*}\Big(\big[\,\overline{D_{ij}}\,\big]\cap\big(\mathrm{pr}_5^*\,\alpha\cup\mathrm{pr}_4^*\,\beta\big)\Big).$

Proof. Let us introduce the incidence varieties

$$T=\Big\{(\underline{x},\underline{y},\underline{z},s,t)\in S^nX\times S^{n-i}X\times S^{n-i+j}X\times X\times X\text{ such that }\underline{x}=\underline{y}+is,\ \underline{z}=\underline{y}+jt\Big\}$$
$$T'=\Big\{(\underline{x},\underline{y},\underline{z},s,t)\in S^nX\times S^{n+j}X\times S^{n-i+j}X\times X\times X\text{ such that }\underline{y}=\underline{x}+js=\underline{z}+it\Big\}$$

Let Ω, Ω' be two small neighbourhoods of T and T' and W a neighbourhood of $Z_{n\times(n-i+j)\times 2}$ such that if $(\underline{x},\underline{y},\underline{z},s,t)\in\Omega$ (resp. Ω'), $\underline{y}\in W_{\underline{x},\underline{z},s,t}$. Let $J^{\mathrm{rel}}_{n\times(n-i+j)\times 2}$ be a relative integrable complex structure on W. After shrinking Ω and Ω' if necessary, we can consider two relative structures $J^{\mathrm{rel}}_{n\times(n-i)\times(n-i+j)\times 2}$ and $J^{\mathrm{rel}}_{n\times(n+j)\times(n-i+j)\times 2}$ such that

$$\begin{cases}\forall(\underline{x},\underline{y},\underline{z},s,t)\in\Omega, & J^{\mathrm{rel}}_{n\times(n-i)\times(n-i+j)\times 2,\underline{x},\underline{y},\underline{z},s,t}=J^{\mathrm{rel}}_{n\times(n-i+j)\times 2,\underline{x},\underline{z},s,t}\\ \forall(\underline{x},\underline{y},\underline{z},s,t)\in\Omega', & J^{\mathrm{rel}}_{n\times(n+j)\times(n-i+j)\times 2,\underline{x},\underline{y},\underline{z},s,t}=J^{\mathrm{rel}}_{n\times(n-i+j)\times 2,\underline{x},\underline{z},s,t}\end{cases}$$

Let U (resp. U') be the points of Y (resp. Y') lying over Ω (resp. Ω'). We define two maps u and v as follows:

$$u:U\longrightarrow\big(X^{[n]\times[n-i+j]\times[1]\times[1]},J^{\mathrm{rel}}_{n\times(n-i+j)\times 2}\leftarrow 4\text{ times}\big),\quad(\xi,\xi',\xi'',s,t)\longmapsto(\xi,\xi'',s,t),$$
$$v:U'\longrightarrow\big(X^{[n]\times[n-i+j]\times[1]\times[1]},J^{\mathrm{rel}}_{n\times(n-i+j)\times 2}\leftarrow 4\text{ times}\big),\quad(\xi,\xi',\xi'',s,t)\longmapsto(\xi,\xi'',s,t)$$

If we take homeomorphisms between $X^{[n]}\times X^{[n-i]}\times X^{[n-i+j]}\times X^2$, $X^{[n]}\times X^{[n+j]}\times X^{[n-i+j]}\times X^2$, $X^{[n]}\times X^{[n-i+j]}\times X^2$ and $X^{[n]\times[n-i]\times[n-i+j]\times[1]\times[1]}$, $X^{[n]\times[n+j]\times[n-i+j]\times[1]\times[1]}$, $X^{[n]\times[n-i+j]\times[1]\times[1]}$, u and v can be extended to global maps which are in the homotopy class of p_{1345}. As in the integrable case, there is an isomorphism $\phi:C_{ij}\xrightarrow{\ \simeq\ }D_{ij}$ given as follows: if $(\xi,\xi',\xi'',s,t)\in C_{ij}$ with $HC(\xi')=\underline{y}$, $HC(\xi)=\underline{y}+is$, $HC(\xi'')=\underline{y}+jt$, then $\phi(\xi,\xi',\xi'',s,t)=(\xi,\widetilde{\xi},\xi'',t,s)$ where $\widetilde{\xi}$ is defined by $\widetilde{\xi}_{|p}=\xi'_{|p}$ if $p\in\underline{y}$, $p\notin\{s,t\}$, $\widetilde{\xi}_{|s}=\xi_{|s}$ and $\widetilde{\xi}_{|t}=\xi''_{|t}$. All these schemes are considered for the structure $J^{\mathrm{rel}}_{n\times(n-i+j)\times 2,\underline{x},\underline{z},s\cup t}$. Let $\partial C_{ij}=\overline{C_{ij}}\backslash C_{ij}$, $\partial D_{ij}=\overline{D_{ij}}\backslash D_{ij}$ and $S=u(\partial C_{ij})=v(\partial D_{ij})$. We define $\pi:Y'\longrightarrow Y'$ by $\pi(\xi,\xi',\xi'',s,t)=(\xi,\xi',\xi'',t,s)$. We have the following diagram, where all the maps are proper:

$$\begin{tikzcd}[column sep=small] Y\backslash\partial C_{ij}\supseteq C_{ij} \arrow[rr, "\phi", "\simeq"', no head] \arrow[dr, "u"'] & & D_{ij}\subseteq Y'\backslash\partial D_{ij} \arrow[dl, "v\circ\pi"] \\ & X^{[n]\times[n-i+j]\times[1]\times[1]}\backslash S & \end{tikzcd}$$

Thus we obtain in the Borel-Moore homology group $H^{\mathrm{lf}}_{2(2n-i+j+2)}\big(X^{[n]\times[n-i+j]\times[1]\times[1]}\backslash S,\mathbb{Q}\big)$ the equality

$$u_*\big([C_{ij}]\cap(\mathrm{pr}_5^*\,\beta\cup\mathrm{pr}_4^*\,\alpha)\big)=v_*\big([D_{ij}]\cap(\mathrm{pr}_4^*\,\beta\cup\mathrm{pr}_5^*\,\alpha)\big).$$

Since $\dim S\leq 2(2n-i+j+2)-2$, we get

$$p_{1345*}\Big(\big[\,\overline{C_{ij}}\,\big]\cap\big(\mathrm{pr}_5^*\,\beta\cup\mathrm{pr}_4^*\,\alpha\big)\Big)=(-1)^{|\alpha|\,|\beta|}p_{1345*}\Big(\big[\,\overline{D_{ij}}\,\big]\cap\big(\mathrm{pr}_5^*\,\alpha\cup\mathrm{pr}_4^*\,\beta\big)\Big).$$

□

By this lemma, in $\big[\mathfrak{q}_{-i}(\alpha), \mathfrak{q}_j(\beta)\big]$, the terms coming from $\overline{C_{ij}}$ and $\overline{D_{ij}}$ cancel out. It remains to deal with the excess intersection components along the diagonals $Y_{\{s=t\}}$ and $Y'_{\{s=t\}}$. We introduce the locus

$$\Gamma = \Big\{\big(\xi,\, \underline{x}, \xi'', \underline{z},\, s,\, t\big) \in X^{[n]\times[n-i+j]\times[1]\times[1]} \text{ such that } s=t,\ \xi_{|p} = \xi''_{|p} \text{ for } p \neq s$$
$$\text{and } HC(\xi'') = HC(\xi) + (j-i)s \text{ if } j \geq i,\ HC(\xi) = HC(\xi'') + (i-j)s \text{ if } j \leq i\Big\}.$$

Γ contains $u(A \cap B)$ and $v(A' \cap B')$. As before, the dimension count can be done as in the integrable case: $\dim \Gamma < 2(2n-i+j+2)$ if $i \neq j$ and if $i=j$, Γ contains a $2(2n+2)$-dimensional component, namely $\Delta_{X^{[n]}} \times \Delta_X$. All other components have lower dimensions. Thus, if $i \neq j$, $p_{1345*}(\iota_* R) = 0$ and $p_{1345*}(\iota'_* R') = 0$ since they are supported in Γ and have degree $2(2n-i+j+2)$. If $i=j$, then $p_{1345*}(\iota_* R)$ and $p_{1345*}(\iota'_* R')$ are proportional to the fundamental class of $\Delta_{X^{[n]}} \times \Delta_X$. Now $p_{45*}\big(\big[\Delta_{X^{[n]}} \times \Delta_X\big] \cap \big(\mathrm{pr}_4^*\, \alpha \cup \mathrm{pr}_5^*\, \beta\big)\big) = \int_X \alpha\beta.\big[\Delta_{X^{[n]}}\big]$ and we obtain $\big[\mathfrak{q}_{-i}(\alpha), \mathfrak{q}_i(\beta)\big] = \mu \int_X \alpha\beta.\,\mathrm{id}$ where $\mu \in \mathbb{Q}$. The computation of the multiplicity μ is a local problem on X which is solved in [Gro], [El-St]. It turns out that $\mu = -i$. □

Remark 4.7. The proof remains quite similar for $i > 0$, $j > 0$. There is no excess term in this case. Indeed, $Y = X^{[n]\times[n+i]\times[n+i+j]\times[1]\times[1]}$, $\Gamma = X^{[n+i+j,n]} \subseteq X^{[n]\times[n+i]\times[n+i+j]\times[1]\times[1]}$ and $\dim \Gamma = 2(2n+i+j+1) < 2(2n+i+j+2)$.

Theorem 4.5 gives a representation in $\mathbb{H} := \bigoplus_{n\in\mathbb{N}} H^*(X^{[n]}, \mathbb{Q})$ of the Heisenberg super-algebra $\mathcal{H}(H^*(X,\mathbb{Q}))$.

Proposition 4.8. *$\mathbb{H}$ is an irreducible $\mathcal{H}(H^*(X,\mathbb{Q}))$-module generated by the vector 1.*

This a consequence of Theorem 4.5 and Göttsche's formula (Theorem 3.9), as shown by Nakajima [Na].

5. Tautological bundles

5.1. **Construction of the tautological bundles.** Our aim in this section is to associate to any complex vector bundle E on an almost-complex compact fourfold X a collection of complex vector bundles $E^{[n]}$ on $X^{[n]}$ which generalize the tautological bundles already known in the algebraic context. The vector bundles $E^{[n]}$ are constructed using an auxiliary relative holomorphic structure on E. However, the classes $E^{[n]}$ in $K(X^{[n]})$ are canonical. Finally, we compare the classes $E^{[n]}$ and $E^{[n+1]}$ in $K(X^{[n]})$ and $K(X^{[n+1]})$ through the incidence variety $X^{[n+1,n]}$.

In the classical case, let E be a algebraic vector bundle on an algebraic surface X. For $n \in \mathbb{N}$, let $p\colon X^{[n]} \times X \longrightarrow X^{[n]}$ and $q\colon X^{[n]} \times X \longrightarrow X$ be the two projections and let $Y_n \subseteq X^{[n]} \times X$ be the incidence locus. Then $p_{|Y_n}\colon Y_n \longrightarrow X^{[n]}$ is finite. The tautological vector bundle $E^{[n]}$ is defined by $E^{[n]} = p_*\big(q^*E \otimes \mathcal{O}_{Y_n}\big)$ and satisfies: for all ξ in $X^{[n]}$, $E^{[n]}_{|\xi} = H^0\big(\xi, i_\xi^* E\big)$. Our first aim is to generalize this construction in the almost-complex case.

Let (X,J) be an almost-complex compact fourfold, $Z_n\subseteq S^nX\times X$ the incidence locus, W a small neighbourhood of Z_n and J_n^{rel} a relative integrable structure on W. The fibers of $\text{pr}_1:W\longrightarrow S^nX$ are smooth analytic sets. We endow W with the sheaf $\mathcal{A}_W$ of continuous functions which are smooth on the fibers of pr_1. We can consider the sheaf $\mathcal{A}^{0,1}_{W,\text{rel}}$ of relative $(0,1)$-forms on W. There exists a relative $\overline{\partial}$-operator $\overline{\partial}^{\,\text{rel}}:\mathcal{A}_W\longrightarrow\mathcal{A}^{0,1}_{W,\text{rel}}$ which induces for each $\underline{x}\in S^nX$ the usual operator $\overline{\partial}:\mathcal{A}_{W_{\underline{x}}}\longrightarrow\mathcal{A}^{0,1}_{W_{\underline{x}}}$ given by the complex structure $J^{\text{rel}}_{n,\underline{x}}$ on $W_{\underline{x}}$.

Definition 5.1. Let E be a complex vector bundle on X.

(i) A *relative connection* $\overline{\partial}_E^{\,\text{rel}}$ *on* E *compatible with* J_n^{rel} is a $\mathbb{C}$-linear morphism of sheaves $\overline{\partial}_E:\mathcal{A}_W\big(\text{pr}_2^*E\big)\longrightarrow\mathcal{A}_W^{0,1}\big(\text{pr}_2^*E\big)$ satisfying $\overline{\partial}_E^{\,\text{rel}}(\varphi s)=\varphi\,\overline{\partial}_E^{\,\text{rel}}s+\overline{\partial}^{\,\text{rel}}\varphi\otimes s$ for all sections φ of $\mathcal{A}_W$ and s of $\mathcal{A}_W\big(\text{pr}_2^*E\big)$.

(ii) A relative connection $\overline{\partial}_E^{\,\text{rel}}$ is *integrable* if $\big(\overline{\partial}_E^{\,\text{rel}}\big)^2=0$.

If $\overline{\partial}_E^{\,\text{rel}}$ is an integrable connection on E compatible with J_n^{rel}, we can apply the Kozsul-Malgrange integrability theorem with continuous parameters in S^nX (see [Vo 3]). Thus, for every $\underline{x}\in S^nX$, $E_{|W_{\underline{x}}}$ is endowed with the structure of a holomorphic vector bundle over $\big(W_{\underline{x}},J^{\text{rel}}_{n,\underline{x}}\big)$ and this structure varies continuously with $\underline{x}$. Furthermore, $\ker\overline{\partial}_E^{\,\text{rel}}$ is the sheaf of relative holomorphic sections of E. Therefore, there is no difference between relative integrable connections on E compatible with J_n^{rel} and relative holomorphic structures on E compatible with J_n^{rel}.

Taking relative holomorphic coordinates for J_n^{rel}, we can see that relative integrable connections exist on W over small open sets of S^nX. By a partition of unity on S^nX, it is possible to build global ones. The space of holomorphic structures on a complex vector bundle over a ball in $\mathbb{C}^2$ is contractible. Therefore the space of relative holomorphic structures on E compatible with J_n^{rel} is also contractible.

We proceed now to the construction of the tautological bundle $E^{[n]}$ on $X^{[n]}_{J_n^{\text{rel}}}$. Let $\overline{\partial}_E^{\,\text{rel}}$ be a relative holomorphic structure on E adapted to J_n^{rel}. Taking relative holomorphic coordinates, we get a vector bundle $E^{[n]}_{\text{rel}}$ over $W^{[n]}_{\text{rel}}$ satisfying: for each $\underline{x}$ in S^nX, $E^{[n]}_{\text{rel}\,|W^{[n]}_{\underline{x}}}=E^{[n]}_{|W_{\underline{x}}}$, where $E_{|W_{\underline{x}}}$ is endowed with the holomorphic structure given by $\overline{\partial}^{\,\text{rel}}_{E,\underline{x}}$.

Definition 5.2. Let $i:X^{[n]}_{J_n^{\text{rel}}}\longrightarrow W^{[n]}_{\text{rel}}$ be the canonical injection. The complex vector bundle $\big(E^{[n]},J_n^{\text{rel}},\overline{\partial}_E^{\,\text{rel}}\big)$ on $X^{[n]}_{J_n^{\text{rel}}}$ is defined by $E^{[n]}=i^*E^{[n]}_{\text{rel}}$.

In the sequel, we consider the class of $E^{[n]}$ in $K\big(X^{[n]}\big)$, which we prove below to be independent of the structures used in the construction.

Proposition 5.3. *The class of* $E^{[n]}$ *in* $K\big(X^{[n]}\big)$ *is independent of* $\big(J_n^{\text{rel}},\overline{\partial}_E^{\,\text{rel}}\big)$.

Proof. Let $\big(J^{\text{rel}}_{0,n},\overline{\partial}^{\,\text{rel}}_{E,0}\big)$ and $\big(J^{\text{rel}}_{1,n},\overline{\partial}^{\,\text{rel}}_{E,1}\big)$ be two relative holomorphic structures on E, $\big(J^{\text{rel}}_{t,n},\overline{\partial}^{\,\text{rel}}_{E,t}\big)$ be a smooth path between them, and $W^{[n]}_{\text{rel}}$ be the relative Hilbert scheme over $S^nX\times[0,1]$ for

the family $\big(J^{\text{rel}}_{t,n}\big)_{0\le t\le 1}$. There exists a vector bundle $\big(\widetilde{E}^{[n]}_{\text{rel}}, \{J^{\text{rel}}_{t,n}\}_{0\le t\le 1}, \{\overline{\partial}^{\text{rel}}_{E,t}\}_{0\le t\le 1}\big)$ over $W^{[n]}_{\text{rel}}$ such that for all t in $[0,1]$, $\widetilde{E}^{[n]}_{\text{rel}\,|\,W^{[n]}_{\text{rel},t}} = \big(E^{[n]}_{\text{rel}}, J^{\text{rel}}_{t,n}, \overline{\partial}^{\text{rel}}_{E,t}\big)$. If $\mathfrak{X} = \big(X^{[n]}, \{J^{\text{rel}}_{t,n}\}_{0\le t\le 1}\big) \subseteq W^{[n]}_{\text{rel}}$ is the relative Hilbert scheme over $[0,1]$, then $\widetilde{E}^{[n]}_{\text{rel}\,|\,\mathfrak{X}}$ is a complex vector bundle on $\mathfrak{X}$ whose restriction to $\mathfrak{X}_t$ is $\big(E^{[n]}, J^{\text{rel}}_{t,n}, \overline{\partial}^{\text{rel}}_{E,t}\big)$. Now $\mathfrak{X}$ is topologically trivial over $[0,1]$ by Proposition 3.10. Since $K(\mathfrak{X}_0\times[0,1])\simeq K(\mathfrak{X}_0)$, we get the result. $\square$

If $\mathbb{T}=X\times\mathbb{C}$ is the trivial complex line bundle on X, the tautological bundles $\mathbb{T}^{[n]}$ already convey geometric informations on $X^{[n]}$. Let $\partial X^{[n]}\subseteq X^{[n]}$ be the inverse image of the big diagonal of S^nX by the Hilbert-Chow morphism. We have $\dim \partial X^{[n]} = 4n-2$ and $H_{4n-2}(\partial X^{[n]},\mathbb{Z})\simeq\mathbb{Z}$ (this can be proved as in Lemma 4.2).

Lemma 5.4. *$c_1(\mathbb{T}^{[n]}) = -\frac{1}{2}\, PD^{-1}\big([\partial X^{[n]}]\big)$ in $H^2\big(X^{[n]},\mathbb{Q}\big)$.*

Proof. Let $U = \big\{(x_1,\dots,x_n)\in X^{[n]}$ such that for all (i,j) with $i\neq j$, $x_i\neq x_j\big\}$. Then $X^{[n]}\setminus\partial X^{[n]}$ is canonically isomorphic to $U/\mathfrak{S}_n$. If $\sigma: U\longrightarrow X^{[n]}\setminus\partial X^{[n]}$ is the associated quotient map, $\sigma^*\mathbb{T}^{[n]}\simeq\bigoplus_{i=1}^{n}\text{pr}_i^*\,\mathbb{T}$, so that $\sigma^*\mathbb{T}^{[n]}$ is trivial. Since σ is a finite covering map, $c_1(\mathbb{T}^{[n]})_{|X^{[n]}\setminus\partial X^{[n]}}$ is a torsion class, so it is zero in $H^2\big(X^{[n]}\setminus\partial X^{[n]},\mathbb{Q}\big)$. This implies that $c_1(\mathbb{T}^{[n]}) = \mu\, PD^{-1}\big([\partial X^{[n]}]\big)$ where $\mu\in\mathbb{Q}$. To compute μ, we argue locally on S^nX around a point in the stratum

$$S=\big\{\underline{x}\in S^nX \text{ such that } x_i\neq x_j \text{ except for one pair } \{i,j\}\big\}.$$

This reduces the computation to the case $n=2$. Then $U^{[2]} = Bl_\Delta(U\times U)/\mathbb{Z}_2$, where $U\subseteq X$ is endowed with an integrable complex structure and Δ is the diagonal of U. If $E\subseteq Bl_\Delta(U\times U)$ is the exceptional divisor and $\pi: Bl_\Delta(U\times U)\longrightarrow U^{[2]}$ is the projection, then $\pi^*\big([\partial U^{[2]}]\big)=2[E]$ and $\pi^*c_1(\mathbb{T}^{[2]}) = c_1(\pi^*\mathbb{T}^{[2]}) = c_1\big(\mathcal{O}(-E)\big) = -[E]$ in $H^2\big(Bl_\Delta(U\times U),\mathbb{Z}\big)$. This gives the value $\mu=-1/2$. $\square$

5.2. **Tautological bundles and incidence varieties.** We want to compare the tautological bundles $E^{[n]}$ and $E^{[n+1]}$ through the incidence variety $X^{[n+1,n]}$. In the integrable case, $X^{[n+1,n]}$ is smooth. If $D\subseteq X^{[n+1,n]}$ is the divisor $\overline{Z}_1$ (see (3)), we have an exact sequence (see [Da], [Le]):

$$0\longrightarrow \rho^*E\otimes\mathcal{O}_{X^{[n+1,n]}}(-D)\longrightarrow \nu^*E^{[n+1]}\longrightarrow\lambda^*E^{[n]}\longrightarrow 0, \tag{5}$$

where $\lambda: X^{[n+1,n]}\longrightarrow X^{[n]}$, $\nu: X^{[n+1,n]}\longrightarrow X^{[n+1]}$ and $\rho: X^{[n+1,n]}\longrightarrow X$ are the two natural projections and the residual map.

In the almost-complex case, $X^{[n+1,n]}$ is a topological manifold of dimension $4n+4$. If we choose a relative integrable structure J^{rel}_{n+1} with additional properties as given in [Vo 1], $X^{[n+1,\,n]}$ can be endowed with a differentiable structure, but we will not need it here.

Let J_n^{rel} and J_{n+1}^{rel} be two relative integrable structures in small neighbourhoods of Z_n and Z_{n+1}. We extend them to relative structures $\check{J}_n^{\mathrm{rel}}$ and $\check{J}_{n+1}^{\mathrm{rel}}$ in small neighbourhoods of $Z_{n\times(n+1)}$. Then $\big(X^{[n]\times[n+1]}, \check{J}_n^{\mathrm{rel}}, \check{J}_{n+1}^{\mathrm{rel}}\big) = X^{[n]}_{J_n^{\mathrm{rel}}} \times X^{[n+1]}_{J_{n+1}^{\mathrm{rel}}}$. If $J^{\mathrm{rel}}_{n\times(n+1)}$ is a relative integrable structure in a small neighbourhood of $Z_{n\times(n+1)}$ and $J^{\mathrm{rel}}_{n\times 1}$ is defined by $J^{\mathrm{rel}}_{n\times 1,\underline{x},p} = J^{\mathrm{rel}}_{n\times(n+1),\underline{x},\underline{x}\cup p}$, then we have a diagram:

$$\begin{tikzcd}
 & & X^{[n+1]}_{J^{\mathrm{rel}}_{n+1}} \\
X^{[n+1,n]}_{J^{\mathrm{rel}}_{n\times 1}} \arrow[r, hook] \arrow[urr, bend left, "\nu"] \arrow[drr, bend right, "\lambda"'] & \big(X^{[n]\times[n+1]}, J^{\mathrm{rel}}_{n\times(n+1)}, J^{\mathrm{rel}}_{n\times(n+1)}\big) \arrow[r, "\Phi", "\simeq"'] & \big(X^{[n]\times[n+1]}, \check{J}_n^{\mathrm{rel}}, \check{J}_{n+1}^{\mathrm{rel}}\big) \arrow[u, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] \\
 & & X^{[n]}_{J^{\mathrm{rel}}_n}
\end{tikzcd}$$

where Φ is a homeomorphism uniquely determined up to homotopy. We will denote by D the inverse image of the incidence locus of $S^nX \times X$ by the map $X^{[n+1,n]} \longrightarrow S^nX \times X$, so that $D = \overline{Z}_1$ where Z_1 is defined by (3). The cycle D has a fundamental homology class in $H_{4n+2}\big(X^{[n+1,n]}, \mathbb{Z}\big)$. Furthermore, there exists a unique complex line bundle F on $X^{[n+1,n]}$ such that $PD\big(c_1(F)\big) = -[D]$.

Proposition 5.5. *In $K\big(X^{[n+1,n]}\big)$, the following identity holds: $\nu^*E^{[n+1]} = \lambda^*E^{[n]} + \rho^*E \otimes F$.*

Proof. Let $\overline{\partial}^{\mathrm{rel}}_{E,n\times 1}$, $\overline{\partial}^{\mathrm{rel}}_{E,n}$ and $\overline{\partial}^{\mathrm{rel}}_{E,n+1}$ be relative holomorphic structures on E compatible with $J^{\mathrm{rel}}_{n\times 1}$, J^{rel}_n and J^{rel}_{n+1}. For each $(\underline{x}, p) \in S^nX \times X$, we consider the exact sequence (5) on $\big(W_{\underline{x},p}, J^{\mathrm{rel}}_{n\times 1,\underline{x},p}\big)$ for the holomorphic vector bundle $\big(E_{|W_{\underline{x},p}}, \overline{\partial}^{\mathrm{rel}}_{E,n\times 1,\underline{x},p}\big)$. Putting these exact sequences in families over $S^nX \times X$, and restricting it to $X^{[n+1,n]}$, we get an exact sequence $0 \longrightarrow \rho^*E \otimes G \longrightarrow A \longrightarrow B \longrightarrow 0$, where G is a complex line bundle on $X^{[n+1,n]}$ and A and B are two vector bundles on $X^{[n+1,n]}$ such that for all $(\underline{x}, p)$ in $S^nX \times X$:

$$A_{|\,\xi,\xi',\underline{x},p} = \Big(E^{[n+1]}_{|\,\xi'}, \overline{\partial}^{\mathrm{rel}}_{E,n\times 1,\underline{x},p}, J^{\mathrm{rel}}_{n\times 1,\underline{x},p}\Big), \quad B_{|\,\xi,\xi',\underline{x},p} = \Big(E^{[n]}_{|\,\xi}, \overline{\partial}^{\mathrm{rel}}_{E,n\times 1,\underline{x},p}, J^{\mathrm{rel}}_{n\times 1,\underline{x},p}\Big). \tag{6}$$

Now, Φ is given by $\Phi(\xi, \xi', \underline{u}, \underline{v}) = \big(\phi_{\underline{u},\underline{v}*}\xi, S^n\phi_{\underline{u},\underline{v}}(\underline{u}), \psi_{\underline{u},\underline{v}*}\xi', S^{n+1}\psi_{\underline{u},\underline{v}*}(\underline{v})\big)$. Thus

$$\nu^*E^{[n+1]}_{|\,\xi,\xi',\underline{x},p} = \Big(E^{[n+1]}_{|\,\psi_{\underline{x},\underline{x}\,\cup\,p*}\xi'}, \overline{\partial}^{\mathrm{rel}}_{E,n+1,S^{n+1}\psi_{\underline{x},\underline{x}\,\cup\,p}(\underline{x}\cup p)}, J^{\mathrm{rel}}_{n+1,S^{n+1}\psi_{\underline{x},\underline{x}\,\cup\,p}(\underline{x}\cup p)}\Big),$$
$$\lambda^*E^{[n]}_{|\,\xi,\xi',\underline{x},x} = \Big(E^{[n]}_{|\,\phi_{\underline{x},\underline{x}\,\cup\,p*}\xi}, \overline{\partial}^{\mathrm{rel}}_{E,n,S^n\phi_{\underline{x},\underline{x}\,\cup\,p}(\underline{x})}, J^{\mathrm{rel}}_{n,S^n\phi_{\underline{x},\underline{x}\,\cup\,p}(\underline{x})}\Big).$$

As in Proposition 5.3, the classes A and B in $K\big(X^{[n+1,n]}\big)$ are independent of the structures used to define them. If $J^{\mathrm{rel}}_{n\times(n+1)} = \check{J}^{\mathrm{rel}}_{n+1}$ and for all $(\underline{x}, p)$ in $S^nX \times X$, $\overline{\partial}^{\mathrm{rel}}_{E,n\times 1,\underline{x},p} = \overline{\partial}^{\mathrm{rel}}_{E,n\times 1,\underline{x}\,\cup\,p}$, we can

take $\psi_{\underline{u},\underline{v}} = \mathrm{id}$ in a neighbourhood of $\underline{v}$. Thus $A = \nu^* E^{[n+1]}$. On the other way, if $J^{\mathrm{rel}}_{n\times(n+1)} = \check{J}^{\mathrm{rel}}_n$ and for all $(\underline{x}, p)$ in $S^n X \times X$, $\overline{\partial}^{\mathrm{rel}}_{E,n\times 1,\underline{x},p} = \overline{\partial}^{\mathrm{rel}}_{E,n,\underline{x}}$ in a neighbourhood of $\underline{x}$, we can take $\phi_{\underline{u},\underline{v}} = \mathrm{id}$ in a neighbourhood of $\underline{u}$. Thus $B = \lambda^* E^{[n]}$. This proves that $\nu^* E^{[n+1]} - \lambda^* E^{[n]} = \rho^* E \otimes G$ in $K\big(X^{[n+1,n]}\big)$. If $\mathbb{T}$ is the trivial complex line bundle on X, $\nu^*\mathbb{T}^{[n+1]} \simeq \lambda^*\mathbb{T}^{[n]} \oplus \rho^*\mathbb{T}$ on $X^{[n+1,n]} \backslash D$. Thus G is trivial outside D. This yields $PD(c_1(G)) = \mu[D]$, where $\mu \in \mathbb{Q}$ and the computation of μ is local, as in Lemma 5.4, so that $\mu = -1$. □

If X is a projective surface, the subring of $H^*(X^{[n]}, \mathbb{Q})$ generated by the classes $\mathrm{ch}_k(E^{[n]})$ (where E runs through all the algebraic vector bundles on X) is called the *ring of algebraic classes of* $X^{[n]}$. If (X, J) is an almost-complex compact fourfold, we can in the same manner consider the subring of $H^*(X^{[n]}, \mathbb{Q})$ generated by the classes $\mathrm{ch}_k(E^{[n]})$, where E runs through all the complex vector bundles on X. If X is projective, this ring is much bigger than the ring of the algebraic classes. In a forthcoming paper, we will show that it is indeed equal to $H^*(X^{[n]}, \mathbb{Q})$ if X is a symplectic compact fourfold satisfying $b_1(X) = 0$, and we will describe the ring structure of $H^*(X^{[n]}, \mathbb{Q})$.

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